SOLUTIONS OF EXERCISES.

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If the angular elevation of the summits of two spires (which appear in a straight line) is α, and the angular depressions of their reflection in a lake, h feet below the point of observation, are β and γ, then the horizontal distance between the spires is

$$2h \cos^2\alpha \sin(\beta-\gamma) \operatorname{cosec}(\beta-\alpha) \operatorname{cosec}(\gamma-\alpha).$$

[*Yale.*]

SOLUTION.

Call the point of observation P, the far summit B, the near A; the reflection of the former C, of the latter D; then, in PDA,

$$\begin{aligned} PA &= PD \sin 2\beta \operatorname{cosec}(\beta-\alpha) \\ &= h \operatorname{cosec}\beta \sin 2\beta \operatorname{cosec}(\beta-\alpha) = 2h \cos\beta \operatorname{cosec}(\beta-\alpha). \end{aligned}$$

Similarly, $PB = 2h \cos\gamma \operatorname{cosec}(\gamma-\alpha)$.

$$\begin{aligned} \therefore AB &= 2h\,[\cos\gamma \operatorname{cosec}(\gamma-\alpha) - \cos\beta \operatorname{cosec}(\beta-\alpha)] \\ &= 2h \cos\alpha \sin(\beta-\gamma) \operatorname{cosec}(\beta-\alpha) \operatorname{cosec}(\gamma-\alpha). \end{aligned}$$

Hence the horizontal projection of AB is

$$2h \cos^2\alpha \sin(\beta-\gamma) \operatorname{cosec}(\beta-\alpha) \operatorname{cosec}(\gamma-\alpha).$$

[*T. U. Taylor.*]

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A regular tetrahedron and a regular octahedron are inscribed in the same sphere; compare the radii of the spheres which can be inscribed in the two solids. [*Yale.*]

SOLUTION.

The radius of the circumscribed sphere is three-fourths the altitude of the tetrahedron and the radius of the inscribed sphere is one-fourth this altitude. Hence the radius of the sphere inscribed in the tetrahedron is $\frac{1}{3}R$, where R is the radius of the sphere. A section of the octahedron through two vertices

and the mid points of two opposite edges gives a lozenge whose diagonals are $2R$ and $R\sqrt{2}$. The perpendicular from the centre on one of these sides is the radius required and is easily found to be $\frac{1}{3}R\sqrt{3}$. The required ratio is, therefore, $1 : \sqrt{3}$. [*T. U. Taylor.*]

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R being the radius of the circle circumscribed about a triangle, r the radius of the circle inscribed in it, and s the half sum of the sides of the triangle, the radii of the escribed circles are the roots of the equation

$$(x^2 + s^2)(x - r) = 4Rx^2.$$

[*Yale.*]

SOLUTION.

The equation may be written

$$x^3 - x^2(4R + r) + xs^2 - rs^2 = 0.$$

Call the roots r_a, r_b, r_c. It is sufficient to show that

$$r_a + r_b + r_c = 4R + r,$$
$$r_ar_b + r_ar_c + r_br_c = s^2,$$
$$r_ar_br_c = rs^2,$$

relations which are easily seen to be true if r_a, r_b, r_c are the radii of the escribed circles. [*T. U. Taylor.*]

EXERCISES.

308

Let points represent complex quantities in the usual way. Show that the quartic whose zeros are any four cotangential points on a fixed circular cubic, has a fixed Jacobian. [*F. Morley.*]

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A horizontal beam of length $2a$, supported at each end, has a load in the form of an inverted parabola symmetrical with respect to the vertical line through the centre of beam. If the vertex of the parabola is b above beam, and if the load is a unit's thickness and has a heaviness unity, the deflection of the beam due to the parabolic load is $\dfrac{61a^4b}{360EI}$. [*T. U. Taylor.*]